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Ascent Sequences and Fibonacci Numbers

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Abstract. An *ascent sequence* is one consisting of non-negative integers in which the size of each letter is restricted by the number of ascents preceding it in the sequence. Ascent sequences have recently been shown to be related to (2+2)-free posets and a variety of other combinatorial structures. Let F_n denote the Fibonacci sequence given by the recurrence $F_n = F_{n-1} + F_{n-2}$ if $n \geq 2$, with $F_0 = 0$ and $F_1 = 1$. In this paper, we draw connections between ascent sequences and the Fibonacci numbers by showing that several pattern-avoidance classes of ascent sequences are enumerated by either F_{n+1} or F_{2n-1}. We make use of both algebraic and combinatorial methods to establish our results. In one of the apparently more difficult cases, we make use of the *kernel method* to solve a functional equation and thus determine the distribution of some statistics on the avoidance class in question. In two other cases, we adapt the *scanning-elements algorithm*, a technique which has been used in the enumeration of certain classes of pattern-avoiding permutations, to the comparable problem concerning pattern-avoiding ascent sequences.

1. Introduction

An *ascent* in a sequence $x_1x_2\cdots x_k$ is a place $j \geq 1$ such that $x_j < x_{j+1}$. An *ascent sequence* $x_1x_2\cdots x_n$ is one consisting of non-negative integers satisfying $x_1 = 0$ and for all i with $1 < i \leq n$,

$$x_i \leq \mathrm{asc}(x_1x_2\cdots x_{i-1}) + 1,$$

where $\mathrm{asc}(x_1x_2\cdots x_k)$ is the number of ascents in the sequence $x_1x_2\cdots x_k$. An example of such a sequence is 01013101542, whereas 0012042 is not since 4 exceeds $\mathrm{asc}(00120) + 1 = 3$. Ascent sequences were first studied in a paper by Bousquet-Mélou, Claesson, Dukes, and Kitaev [3], where they were shown to have the same cardinality as the (2+2)-free posets of the same size and the generating function was determined. Since then they have been studied in a series of papers by various authors where connections were made to many other combinatorial structures, including certain integer matrices, set partitions, and permutations. See, for example, [4, 5, 9] as well as [8, Section 3.2.2] for further information.

Let F_n denote the n-th Fibonacci number defined by $F_n = F_{n-1} + F_{n-2}$ if $n \geq 2$, with $F_0 = 0$ and $F_1 = 1$. See A000045 in [11]. In this paper, we draw connections between ascent sequences and Fibonacci numbers by showing that certain avoidance classes of ascent sequences, each involving two patterns, are enumerated by F_{n+1} or F_{2n-1}. This extends recent work in [6], where the problem of avoidance of a single pattern was considered.

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We will refer to a sequence of non-negative integers, where repetitions are allowed, as a *pattern*. Let $\pi = \pi_1\pi_2\cdots\pi_n$ be an ascent sequence and $\tau = \tau_1\tau_2\cdots\tau_m$ be a pattern. Then we say that π *contains* τ if π has a subsequence that is order isomorphic to τ, that is, there is a subsequence $\pi_{f(1)}, \pi_{f(2)}, \ldots, \pi_{f(m)}$, where $1 \le f(1) < f(2) < \cdots < f(m) \le n$, such that for all $1 \le i, j \le m$, we have $\pi_{f(i)} < \pi_{f(j)}$ if and only if $\tau_i < \tau_j$ and $\pi_{f(i)} > \pi_{f(j)}$ if and only if $\tau_i > \tau_j$. Otherwise, π is said to *avoid* τ. For example, the ascent sequence 010103422451 has three occurrences of the pattern 110, namely, the subsequences 110, 441, and 221, but avoids the pattern 201. Note that within an occurrence of a pattern τ, letters corresponding to equal letters in τ must be equal within the occurrence.

To be consistent with the usual notation for ascent sequences which contains 0's, we will write patterns for ascent sequences using non-negative integers as in [6], though patterns for other structures like permutations are traditionally written with positive integers. Thus, the usual patterns will have different names here; for example, 123 becomes 012 and 211 becomes 100. Given a pair of patterns τ and ρ, let $\mathcal{S}_n(\tau,\rho)$ denote the set of ascent sequences of length n avoiding both τ and ρ, and let $A_n(\tau,\rho)$ be the number of such sequences. By the (i, j)-avoidance class corresponding to a given sequence $(a_n)_{n\ge0}$ of non-negative integers, we will mean the (Wilf) equivalence class comprising all sets $\mathcal{S}_n(\tau,\rho)$ such that $|\mathcal{S}_n(\tau,\rho)| = a_n$ for all $n \ge 0$, where i and j denote the respective lengths of τ and ρ.

Below, we determine all members of the (3, 3)- and (3, 4)-avoidance classes of ascent sequences corresponding to Fibonacci numbers. Furthermore, in several cases, we establish the distribution for the number of ascents on the set $\mathcal{S}_n(\tau,\rho)$ in question. We make use of both algebraic and combinatorial arguments to establish our results. For the cases of avoiding (110, 0122) and (100, 0121), we adapt the scanning-elements algorithm [7], a technique which has been used in the study of permutation patterns, to the ascent sequence structure. This enables one to write a system of equations involving various generating functions which can be solved to determine the generating function for $A_n(\tau,\rho)$ in each case.

In the final section, we consider the pattern pair (021, 0122) (or, equivalently, the pair (021, 120) via bijection), which seems to be more difficult to enumerate. To do so, we first refine the numbers $A_n(021, 0122)$ according to three statistics on $\mathcal{S}_n(021, 0122)$ which record the number of ascents, the largest letter, and the last letter of a sequence. See the paper by Zeilberger [12] for a discussion of a general strategy of refinement which we are adapting to this case in determining an unknown integer sequence. Next, we determine a three-parameter functional equation satisfied by the generating function, which we denote by $f(x; u, v, w)$, for the joint distribution polynomial of the aforementioned statistics. We then use the *kernel method* (see [1]) to solve this functional equation and find $f(x; 1, 1, 1)$, which will imply $A_n(021, 0122) = F_{2n-1}$. Furthermore, one may determine the full expression for $f(x; u, v, w)$ and various distributions on $\mathcal{S}_n(021, 0122)$ involving combinations of the aforementioned statistics may then be obtained from it, leading to various refinements of the numbers F_{2n-1}.

2. Avoiding Two Patterns of Length Three

The following proposition gives the two members of the (3, 3)-avoidance class for ascent sequences corresponding to the Fibonacci number F_{n+1}.

Proposition 2.1. *If $n \ge 0$, then $A_n(000, 001) = A_n(000, 010) = F_{n+1}$.*

Proof. We shall show that $A_n(000, 001) = F_{n+1}$ by induction on n, the $n = 1$ and $n = 2$ cases clear. If $n \ge 3$, then there are F_{n-1} members of $\mathcal{S}_n(000, 001)$ in which the largest letter ℓ appears twice, as seen upon removing both copies of the letter ℓ, which are adjacent and must directly follow the first occurrence of the letter $\ell - 1$. Note that $n \ge 3$ implies $\ell \ge 1$ since 000 is not allowed. Similarly, there are F_n members of $\mathcal{S}_n(000, 001)$ in which the largest letter appears once. There are then $F_{n-1} + F_n = F_{n+1}$ members of $\mathcal{S}_n(000, 001)$ in all, which completes the induction. A similar proof shows that there are F_{n+1} members of $\mathcal{S}_n(000, 010)$. □

Recall that
$$\sum_{n\ge0} F_{2n-1}x^n = \frac{1-2x}{1-3x+x^2},$$

where $F_{-1} = 1$. Our next result concerns the $(3,3)$-Wilf equivalence class for ascent sequences corresponding to the number F_{2n-1}.

Proposition 2.2. *If $n \geq 0$, then $A_n(u,v) = F_{2n-1}$ for the following pairs (u,v):*

$$(i)\ (021,101) \quad (iii)\ (101,102)$$
$$(ii)\ (021,120) \quad (iv)\ (101,110).$$

Proof. (i) We will determine more. Let $f(x,q)$ denote the generating function which counts the members of $\mathcal{S}_n(021,101)$ according to the number of ascents, i.e.,

$$f(x,q) = \sum_{n\geq 0}\left(\sum_{\pi\in\mathcal{S}_n(021,101)} q^{\mathrm{asc}(\pi)}\right)x^n.$$

Note that if $\pi \in \mathcal{S}_n(021,101)$ contains at least two distinct letters, then it must have the form

$$\pi = 0^{i_0}1^{i_1}\cdots\ell^{i_\ell}\pi',$$

where $\ell \geq 1$, $i_0, i_1, \ldots, i_\ell > 0$, and π' is possibly empty and starts with 0 if non-empty. Note that π' is itself an ascent sequence on the letters $\{0, \ell+1, \ell+2, \ldots\}$ which avoids 021 and 101. Conversely, an ascent sequence of the form π, with π' as described, is a member of $\mathcal{S}_n(021,101)$. Upon including the empty sequence and sequences of the form 0^n for some $n \geq 1$, we have

$$f(x,q) = 1 + \frac{x}{1-x} + \left(\frac{x}{1-x}\right)\cdot\left(\frac{\frac{qx}{1-x}}{1-\frac{qx}{1-x}}\right)f(x,q) = \frac{1}{1-x} + \frac{qx^2}{(1-x)(1-x-xq)}f(x,q),$$

since each run of letters j in π for $1 \leq j \leq \ell$ contributes one ascent, with ℓ any positive number as n ranges. Solving for $f(x,q)$ gives

$$f(x,q) = \frac{1-(1+q)x}{1-(2+q)x+x^2},$$

and taking $q = 1$ yields the first case.

(ii) This follows from combining Proposition 4.1 and Theorem 4.5 in the last section.

(iii) Let $g(x,q)$ denote the generating function which counts the members of $\mathcal{S}_n(101,102)$ according to the number of ascents. First note that a non-empty ascent sequence π which avoids 101 and 102 is either of the form (i) $\pi = 0\pi'$, where π' contains no 0's, (ii) $\pi = 00\pi'$, where π' may contain 0's, or (iii) $\pi = 01\pi'0\cdots0$, ending in a non-empty run of 0's, where π' contains no 0's. In all cases, note that π' must avoid the patterns 101 and 102. Combining the three cases then gives

$$g(x,q) = 1 + qxg(x,q) + x(g(x,q)-1) + \frac{qx^2}{1-x}(g(x,q)-1).$$

Thus,

$$g(x,q) = \frac{1-2x+(1-q)x^2}{1-(2+q)x+x^2},$$

from which this case follows by taking $q = 1$.

(iv) Let $h(x,q)$ denote the comparable generating function in this case. First note a non-empty ascent sequence π avoiding both 101 and 110 and containing two or more zeros must be of the form $\pi = 01\cdots i\pi'$, where $i \geq 0$ and π' itself is a non-empty ascent sequence on the letters $\{0, i+1, i+2, \ldots\}$ avoiding the patterns. Then considering whether a sequence contains one or more zeros gives

$$h(x,q) = 1 + qxh(x,q) + \frac{x}{1-qx}(h(x,q)-1),$$

which implies

$$h(x,q) = \frac{1-(1+q)x}{1-(1+2q)x+q^2x^2}$$

and completes the proof. □

Let $\mathcal{F}_m$ denote the set of coverings of the numbers $1, 2, \ldots, m$, arranged in a row, by indistinguishable dominos and indistinguishable squares, where pieces do not overlap, a domino is a piece covering two numbers, and a square is a piece covering a single number. The members of $\mathcal{F}_m$ are also called *tilings* or *square-and-domino* arrangements (see [2, Chapter 1]). Note that $|\mathcal{F}_m| = F_{m+1}$ for all m. Furthermore, letting s and d stand for *square* and *domino*, respectively, it is easily seen that members of $\mathcal{F}_m$ correspond uniquely to words in the alphabet $\{d, s\}$ containing k d's and $m-2k$ s's for some k, $0 \le k \le \frac{m}{2}$. In the proof of the following proposition, we will identify the members of $\mathcal{F}_m$ by such words. Furthermore, given $\delta \in \mathcal{F}_{2m}$, let $s(\delta)$ be half the number of squares in δ.

Proposition 2.3. *If $n \ge 1$, then there is a bijection g from $\mathcal{S}_n(021,101)$ to $\mathcal{F}_{2n-2}$ such that $asc(\pi) = s(g(\pi))$ for all $\pi \in \mathcal{S}_n(021,101)$. Therefore, the number of members of $\mathcal{S}_n(021,101)$ having exactly k ascents is $\binom{n-1+k}{2k}$.*

Proof. Suppose $\pi \in \mathcal{S}_n(021,101)$, where $n \ge 1$. Then we may express π as

$$\pi = 0^{i_0}1^{j_1}0^{i_1}2^{j_2}0^{i_2}\cdots\ell^{j_\ell}0^{i_\ell},$$

where $\ell \ge 0$ represents the largest letter of π, $j_1, j_2, \ldots, j_\ell > 0$, $i_0 > 0$, and $i_1, i_2, \ldots, i_\ell \ge 0$. Let

$$g(\pi) = d^{i_0-1}(sd^{i_1}sd^{j_1-1})(sd^{i_2}sd^{j_2-1})\cdots(sd^{i_\ell}sd^{j_\ell-1}).$$

Then $g(\pi) \in \mathcal{F}_{2n-2}$ and it may be verified that the mapping g is a bijection such that $\text{asc}(\pi) = s(g(\pi))$ for all π. The second statement is an easy consequence of the first. □

3. The (3,4)-Classes

In this section, we consider the ascent sequences enumerated by F_{2n-1} which avoid a pattern of length three and another of length four. For a couple of cases, we modify the scanning-elements algorithm (see [7]) used previously on permutations as follows. Let $h_T(x,q|\tau)$ denote the generating function which counts the ascent sequences of length n whose first letters form the word τ and that avoid all the patterns in the set T according to some statistic ρ marked by the variable q, where τ, ρ, and T are given. We may assume τ avoids all the patterns in T, for otherwise the generating function is zero. Scanning the next letter j, we obtain

$$h_T(x,q|\tau) = x^{|\tau|}q^{\rho(\tau)} + \sum_{j=0}^{t} h_T(x,q|\tau j),$$

where $t = \text{asc}(\tau)+1$. In some cases, applying this equation repeatedly leads to a finite system of equations as below which can be solved, and the generating function $h_T(x,q|\tau)$ would be rational in that case. There are limitations to this method though. In some cases, the system is infinite and perhaps one could attempt to solve it using the kernel method, though many times the system cannot be solved in such cases. Furthermore, some sets of patterns T do not permit one to write a meaningful system of equations. Below, we provide a couple of illustrations of this method in the case when T contains two patterns.

We first consider the problem of counting the members of $\mathcal{S}_n(110,0122)$. Let

$$f(x,q) = \sum_{n\ge 0}\left(\sum_{\pi\in\mathcal{S}_n(110,0122)} q^{asc(\pi)}\right)x^n.$$

In order to determine $f(x,q)$, we define $f(x,q|a_1a_2\cdots a_m)$ to be the generating function counting the ascent sequences $\pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n(110,0122)$ according to the number of ascents such that $\pi_1\pi_2\cdots\pi_m = a_1a_2\cdots a_m$.

From the definitions, we have

$$
\begin{aligned}
f(x,q) &= 1 + f(x,q|0),\\
f(x,q|0) &= x + f(x,q|00) + f(x,q|01)\\
&= x + xf(x,q|0) + f(x,q|01),\\
f(x,q|01) &= x^2q + f(x,q|010) + f(x,q|011) + f(x,q|012)\\
&= x^2q + f(x,q|010) + f(x,q|011) + xqf(x,q|01),\\
f(x,q|011) &= x^3q + f(x,q|0111) + f(x,q|0112)\\
&= x^3q + xf(x,q|011) + xqf(x,q|011),\\
f(x,q|010) &= x^3q + f(x,q|0100) + f(x,q|0101) + f(x,q|0102)\\
&= x^3q + xf(x,q|010) + f(x,q|0101) + x^2qf(x,q|01),\\
f(x,q|0101) &= x^4q^2 + f(x,q|01011) + f(x,q|01012) + f(x,q|01013)\\
&= x^4q^2 + xf(x,q|0101) + xqf(x,q|0101) + f(x,q|01013),\\
f(x,q|01013) &= x^5q^3 + f(x,q|010131) + f(x,q|010132) + f(x,q|010134)\\
&= x^5q^3 + x^2qf(x,q|0101) + f(x,q|010132) + xqf(x,q|01013),\\
f(x,q|010132) &= x^6q^3 + f(x,q|0101321) + f(x,q|0101324)\\
&= x^6q^3 + f(x,q|0101321) + xqf(x,q|010132),\\
f(x,q|0101321) &= x^7q^3 + f(x,q|01013211) + f(x,q|01013214)\\
&= x^7q^3 + xf(x,q|0101321) + xqf(x,q|0101321).
\end{aligned}
$$

The reduction of letters in various steps may be explained through bijections. For example, we have $f(x,q|012) = xqf(x,q|01)$, since the map $01\pi_3\pi_4\cdots\pi_{n-1} \mapsto 012\pi'_3\pi'_4\cdots\pi'_{n-1}$, where $\pi'_i = \pi_i + 1$ if $\pi_i \geq 2$ and $\pi'_i = \pi_i$ if $\pi_i = 0, 1$, defines a bijection between the set $\{\alpha \in \mathcal{S}_{n-1}(110, 0122)|\alpha_1\alpha_2 = 01\}$ and the set $\{\alpha \in \mathcal{S}_n(110, 0122)|\alpha_1\alpha_2\alpha_3 = 012\}$. As a second example, to show $f(x,q|010131) = x^2qf(x,q|0101)$, observe that the map $0101\pi_5\pi_5\cdots\pi_{n-2} \mapsto 010131\widehat{\pi}_5\widehat{\pi}_6\cdots\widehat{\pi}_{n-2}$, where $\widehat{\pi}_i = \pi_i + 1$ if $\pi_i \geq 3$ and $\widehat{\pi}_i = \pi_i$ if $\pi_i = 1, 2$, defines a bijection between the set $\{\alpha \in \mathcal{S}_{n-2}(110, 0122)|\alpha_1\alpha_2\alpha_3\alpha_4 = 0101\}$ and the set $\{\alpha \in \mathcal{S}_n(110, 0122)|\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6 = 010131\}$.

Solving the above system of equations (using Maple) then gives

$$f(x,q) = 1 + \frac{x(1-xq)}{(1-xq)^2 - x} = 1 + \sum_{j\geq 0} \frac{x^{j+1}}{(1-xq)^{2j+1}} = 1 + \sum_{j\geq 0}\sum_{i\geq 0} \binom{2j+i}{i} x^{i+j+1} q^i.$$

Hence, we can state the following result.

Theorem 3.1. *The number of ascent sequences in $\mathcal{S}_n(110, 0122)$ is given by F_{2n-1}. Moreover, the number of ascent sequences in $\mathcal{S}_n(110, 0122)$ with exactly k ascents is given by $\binom{2n-2-k}{k}$, $0 \leq k \leq n-1$.*

We next count the members of $\mathcal{S}_n(100, 0121)$. Let

$$g(x,q) = \sum_{n\geq 0}\left(\sum_{\pi\in\mathcal{S}_n(100,0121)} q^{asc(\pi)}\right)x^n.$$

In order to determine $g(x,q)$, we define $g(x,q|a_1a_2\cdots a_m)$ to be the generating function counting the ascent sequences $\pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n(100, 0121)$ according to the number of ascents such that $\pi_1\pi_2\cdots\pi_m = a_1a_2\cdots a_m$.

From the definitions, we have

$$\begin{aligned}
g(x,q) &= 1 + g(x,q|0),\\
g(x,q|0) &= x + xg(x,q|0) + g(x,q|01),\\
g(x,q|01) &= x^2q + g(x,q|010) + xg(x,q|01) + xqg(x,q|01),\\
g(x,q|010) &= x^3q + g(x,q|0101) + g(x,q|0102),\\
g(x,q|0102) &= \frac{x^4q^2}{1-x-xq},\\
g(x,q|0101) &= x^4q^2 + xg(x,q|0101) + g(x,q|01012) + g(x,q|01013),\\
g(x,q|01012) &= x^5q^3 + xg(x,q|01012) + xqg(x,q|01012) + xqg(x,q|01013),\\
g(x,q|01013) &= x^5q^3 + g(x,q|010132) + xg(x,q|01013) + xqg(x,q|01013),\\
g(x,q|010132) &= x^6q^3 + x^2qg(x,q|01012) + g(x,q|0101324),\\
g(x,q|0101324) &= \frac{x^7q^4}{1-x-xq}.
\end{aligned}$$

The reduction of letters may be explained through bijections as in the previous case. Note that $g(x,q|0102) = \frac{x^4q^2}{1-x-xq}$ since the fifth letter (if it occurs) is either a 2 or a 3, with each subsequent letter either equal to or one more than its predecessor with a factor of q introduced each time the latter occurs. Similar reasoning may be used to explain the formula for $g(x,q|0101324)$.

Solving the above system of equations (using Maple) then gives

$$g(x,q) = 1 + \frac{x(1-xq)}{(1-xq)^2 - x} = 1 + \sum_{j\geq 0} \frac{x^{j+1}}{(1-xq)^{2j+1}} = 1 + \sum_{j\geq 0}\sum_{i\geq 0} \binom{2j+i}{i} x^{i+j+1} q^i.$$

Hence, we can state the following result.

Theorem 3.2. *The number of ascent sequences in $\mathcal{S}_n(100,0121)$ is given by F_{2n-1}. Moreover, the number of ascent sequences in $\mathcal{S}_n(100,0121)$ with exactly k ascents is given by $\binom{2n-2-k}{k}$, $0 \leq k \leq n-1$.*

In the following proposition, we give the members of the (3, 4)-Wilf equivalence class for ascent sequences corresponding to the Fibonacci number F_{2n-1}.

Proposition 3.3. *If $n \geq 0$, then $A_n(u,v) = F_{2n-1}$ for the following pairs (u,v):*

(1) (021, 0101)	(2) (021, 0120)	(3) (021, 0122)
(4) (100, 0121)	(5) (101, 0011)	(6) (101, 0102)
(7) (101, 0110)	(8) (101, 0121)	(9) (102, 0001)
(10) (102, 0012)	(11) (102, 0101)	(12) (110, 0101)
(13) (110, 0122)	(14) (120, 0123).	

Proof. Note that cases (13) and (4) were shown above and case (3) is treated in the next section. For (2), it is not hard to show that avoiding (021, 0120) is logically equivalent to avoiding (021, 120), the latter given in Proposition 2.2 above. A modification of the proof given in [10, Theorem 3.3] concerning avoiding the single pattern 0123 yields (14). The remaining cases are easier and may be done using a variety of algebraic and combinatorial arguments, which we leave as exercises for the interested reader. □

Remark: Numerical evidence shows that there are no other members of the (3,4)-Wilf equivalence class corresponding to F_{2n-1}.

4. The Case of Avoiding 021 and 0122

In this section, we seek to determine the joint distribution of some statistics on $\mathcal{S}_n(021, 0122)$, and we find the generating function for this distribution from which explicit formulas may be obtained. Taking all but one of the parameters to be unity will show $A_n(021, 0122) = F_{2n-1}$ if $n \geq 1$ and thus various refinements of the numbers F_{2n-1} are obtained. By the following proposition, we also have $A_n(021, 0120) = F_{2n-1}$.

Proposition 4.1. *If $n \geq 1$, then $A_n(021, 120) = A_n(021, 0122)$.*

Proof. Suppose $\lambda \in \mathcal{S}_n(021, 120)$, where $n \geq 1$. Then λ may be expressed in the form

$$\lambda = \alpha w_1 w_2 \cdots w_\ell,$$

where α is binary, $\ell \geq 0$ ($\ell = 0$ corresponds to when λ is binary), and w_i is a non-empty run of the letter r_i for some $r_i \geq 2$, with $r_1 < r_2 < \cdots < r_\ell$. Conversely, any sequence of this form is seen to belong to $\mathcal{S}_n(021, 120)$. Define the mapping f of $\mathcal{S}_n(021, 120)$ by

$$f(\lambda) = \alpha w_1' w_2' \cdots w_\ell',$$

where w_i' is obtained from w_i by replacing all but the first letter of w_i with 0. It may be verified that f is a bijection between $\mathcal{S}_n(021, 120)$ and $\mathcal{S}_n(021, 0122)$. □

Let $\mathcal{A}_n = \mathcal{S}_n(021, 0122)$. We first refine the set $\mathcal{A}_n$ as follows. Given $n \geq 1$ and $0 \leq s \leq r \leq m < n$, let $\mathcal{A}_{n,m,r,s}$ denote the subset of $\mathcal{A}_n$ whose members have m ascents, largest letter r, and last letter s. For example, we have $\pi = 0110120030 \in \mathcal{A}_{10,4,3,0}$. The numbers $a_{n,m,r,s} = |\mathcal{A}_{n,m,r,s}|$ may be determined as described in the following lemma.

Lemma 4.2. *The array $a_{n,m,r,s}$ can assume non-zero values only when $n \geq 1$ and $0 \leq s \leq r \leq m < n$. It satisfies the conditions $a_{n,0,0,0} = 1$ and $a_{n,m,0,0} = 0$ if $n, m \geq 1$. For $n \geq 2$ and $1 \leq m \leq n-1$, the numbers $a_{n,m,r,s}$ satisfy*

$$a_{n,m,r,0} = \sum_{i=0}^{r} a_{n-1,m,r,i}, \qquad r \geq 1, \tag{1}$$

and

$$a_{n,m,r,s} = \sum_{j=0}^{r-1} \sum_{i=0}^{j} a_{n-1,m-1,j,i}, \qquad r = s \geq 2, \tag{2}$$

with $a_{n,m,r,s} = 0$ if $r > s \geq 1$ and $a_{n,m,1,1} = \binom{n-1}{2m-1}$.

Proof. The first two statements are clear from the definitions. To show (1), we delete the final 0 from $\pi \in \mathcal{A}_{n,m,r,0}$, since it is extraneous concerning a possible occurrence of 021 or 0122, and the resulting ascent sequence belongs to $\bigcup_{i=0}^{r} \mathcal{A}_{n-1,m,r,i}$. Furthermore, note that $\mathcal{A}_{n,m,r,s}$ is empty if $r > s \geq 1$ since we must avoid 021.

So suppose $\pi \in \mathcal{A}_{n,m,s,s}$, where $1 \leq m \leq n-1$ and $s \geq 1$. If $s = 1$, then $\pi \in \mathcal{A}_{n,m,1,1}$ is a binary sequence starting with 0, ending in 1, and containing exactly m ascents, which implies $a_{n,m,1,1} = \binom{n-1}{2m-1}$. If $s \geq 2$, then we remove the s in the last position from $\pi \in \mathcal{A}_{n,m,s,s}$. Note that this is the only occurrence of s since we are to avoid 0122, whence an ascent is lost from π when we remove s. Thus the resulting ascent sequence belongs to $\mathcal{A}_{n-1,m-1,j,i}$ for some i and j with $j \leq s-1$. Summing over all possible i and j gives (2). □

Define the polynomials $A_{n,m,r}(u) = \sum_{s=0}^{r} a_{n,m,r,s} u^s$, where $n \geq 1$ and $0 \leq r \leq m < n$. Note that

$$A_{n,m,0}(u) = \begin{cases} 1, & \text{if } m = 0; \\ 0, & \text{if } m > 0, \end{cases}$$

and

$$A_{n,m,1}(u) = \binom{n-1}{2m} + \binom{n-1}{2m-1}u, \qquad m \geq 1,$$

the latter holding since there are $\binom{n-1}{2m}$ members of $\mathcal{A}_{n,m,1,0}$ and $\binom{n-1}{2m-1}$ members of $\mathcal{A}_{n,m,1,1}$.

Let $A_{n,m}(u,v) = \sum_{r=0}^{m} A_{n,m,r}(u)v^r$, where $0 \leq m < n$. Note that $A_{n,0}(u,v) = 1$ and

$$A_{n,1}(u,v) = A_{n,1,1}(u)v = \binom{n-1}{2}v + (n-1)uv.$$

The polynomials $A_{n,m}(u,v)$ satisfy the following recurrence for $m \geq 2$.

Lemma 4.3. *If $n \geq 3$ and $2 \leq m \leq n-1$, then*

$$A_{n,m}(u,v) = \binom{n-1}{2m-1}uv + A_{n-1,m}(1,v) + \frac{uv}{1-uv}(A_{n-1,m-1}(1,uv) - (uv)^m A_{n-1,m-1}(1,1)). \tag{3}$$

Proof. First note that if $r \geq 2$, then by (1) and (2), we have

$$\begin{aligned} A_{n,m,r}(u) &= \sum_{s=0}^{r} a_{n,m,r,s}u^s = \sum_{i=0}^{r} a_{n-1,m,r,i} + u^r \sum_{j=0}^{r-1}\sum_{i=0}^{j} a_{n-1,m-1,j,i} \\ &= A_{n-1,m,r}(1) + u^r \sum_{j=0}^{r-1} A_{n-1,m-1,j}(1). \end{aligned} \tag{4}$$

By (4), we then have

$$\begin{aligned} A_{n,m}(u,v) &= A_{n,m,0}(u) + A_{n,m,1}(u)v + \sum_{r=2}^{m} A_{n,m,r}(u)v^r \\ &= A_{n,m,1}(u)v + \sum_{r=2}^{m} A_{n-1,m,r}(1)v^r + \sum_{r=2}^{m}(uv)^r \sum_{j=0}^{r-1} A_{n-1,m-1,j}(1) \\ &= \binom{n-1}{2m}v + \binom{n-1}{2m-1}uv + (A_{n-1,m}(1,v) - A_{n-1,m,1}(1)v) \\ &\quad + \sum_{j=0}^{m-1} A_{n-1,m-1,j}(1) \sum_{r=j+1}^{m}(uv)^r \\ &= \binom{n-1}{2m}v + \binom{n-1}{2m-1}uv + A_{n-1,m}(1,v) - \left(\binom{n-2}{2m} + \binom{n-2}{2m-1}\right)v \\ &\quad + \sum_{j=0}^{m-1} A_{n-1,m-1,j}(1)\frac{(uv)^{j+1} - (uv)^{m+1}}{1-uv} \\ &= \binom{n-1}{2m-1}uv + A_{n-1,m}(1,v) + \frac{uv}{1-uv}(A_{n-1,m-1}(1,uv) - (uv)^m A_{n-1,m-1}(1,1)), \end{aligned}$$

which completes the proof. □

If $n \geq 1$, then let $A_n(u,v,w) = \sum_{m=0}^{n-1} A_{n,m}(u,v)w^m$ and let

$$f(x;u,v,w) = \sum_{n\geq 1} A_n(u,v,w)x^n.$$

Then f satisfies the following functional equation.

Lemma 4.4. *We have*

$$f(x;u,v,w) = \frac{x-(1+uvw)x^2}{1-x} + \frac{uvwx^2}{(1-x)^2-wx^2} + xf(x;1,v,w) + \frac{uvwx}{1-uv}(f(x;1,uv,w) - uvf(x;1,1,uvw)). \tag{5}$$

Proof. If $n \geq 3$, then by (3), we have

$$\begin{aligned} A_n(u,v,w) &= A_{n,0}(u,v) + A_{n,1}(u,v)w + \sum_{m=2}^{n-1} A_{n,m}(u,v)w^m \\ &= 1 + \binom{n-1}{2}vw + (n-1)uvw + uv\sum_{m=2}^{n-1}\binom{n-1}{2m-1}w^m \\ &\quad + (A_{n-1}(1,v,w) - A_{n-1,1}(1,v)w - 1) + \frac{uvw}{1-uv}(A_{n-1}(1,uv,w)-1) \\ &\quad - \frac{u^2v^2w}{1-uv}(A_{n-1}(1,1,uvw)-1) \\ &= (n-2)uvw + uv\left(\sum_{m=1}^{n-1}\binom{n-1}{2m-1}w^m - (n-1)w\right) + A_{n-1}(1,v,w) \\ &\quad + \frac{uvw}{1-uv}(A_{n-1}(1,uv,w) - uvA_{n-1}(1,1,uvw)) \\ &= -uvw + \frac{uv\sqrt{w}}{2}\left((1+\sqrt{w})^{n-1} - (1-\sqrt{w})^{n-1}\right) + A_{n-1}(1,v,w) \\ &\quad + \frac{uvw}{1-uv}(A_{n-1}(1,uv,w) - uvA_{n-1}(1,1,uvw)). \end{aligned} \tag{6}$$

Note that (6) also holds for $n = 2$ since $A_1(u,v,w) = 1$. Multiplying (6) by x^n and summing over $n \geq 2$ implies

$$\begin{aligned} f(x;u,v,w) - x &= -\frac{uvwx^2}{1-x} + \frac{uv\sqrt{w}x}{2}\left(\frac{(1+\sqrt{w})x}{1-(1+\sqrt{w})x} - \frac{(1-\sqrt{w})x}{1-(1-\sqrt{w})x}\right) \\ &\quad + xf(x;1,v,w) + \frac{uvwx}{1-uv}(f(x;1,uv,w) - uvf(x;1,1,uvw)), \end{aligned}$$

which gives (5). □

Theorem 4.5. *We have*

$$f(x;1,1,w) = \frac{x-x^2}{1-(2+w)x+x^2}, \tag{7}$$

and thus the number of ascent sequences in $\mathcal{A}_n$ with exactly k ascents is given by $\binom{n-1+k}{2k}$. In particular, we have $|\mathcal{A}_n| = F_{2n-1}$ if $n \geq 1$.

Proof. We will only show (7), the second and third statements being easy consequences of it. Setting $u = 1$ in (5) gives

$$\left(1 - x - \frac{vwx}{1-v}\right)f(x;1,v,w) = \frac{x-(1+vw)x^2}{1-x} + \frac{vwx^2}{(1-x)^2-wx^2} - \frac{v^2wx}{1-v}f(x;1,1,vw). \tag{8}$$

To solve (8), we use the *kernel method* (see [1]). Setting the coefficient of $f(x;1,v,w)$ on the left-hand side of (8) equal to zero, and solving for $v = v_o$ in terms of x and w gives $1 - x = \frac{v_owx}{1-v_o}$, i.e., $v_o = \frac{1-x}{1-x+wx}$. Setting $v = v_o$

in (8) then gives

$$v_o(1-x)f(x;1,1,v_ow) = x - \frac{v_owx^2}{1-x} + \frac{v_owx^2}{(1-x)^2-wx^2} = x - x(1-v_o) + \frac{v_owx^2}{(1-x)^2-wx^2}$$
$$= v_ox\left(1+\frac{wx}{(1-x)^2-wx^2}\right)$$

and thus

$$f(x;1,1,v_ow) = \frac{x}{1-x} + \frac{wx^2}{(1-x)((1-x)^2-wx^2)}. \quad (9)$$

Let us replace the argument v_ow appearing on the left-hand side of (9) with t, i.e., $w = \frac{t(1-x)}{1-x-tx}$. Then (9) may be rewritten as

$$f(x;1,1,t) = \frac{x}{1-x} + \frac{tx^2}{(1-x-tx)\left((1-x)^2-\frac{tx^2(1-x)}{1-x-tx}\right)} = \frac{x}{1-x} + \frac{tx^2}{(1-x)((1-x)(1-x-tx)-tx^2)}$$
$$= \frac{x-x^2}{1-(2+t)x+x^2},$$

which gives (7). □

Using (7) and (8), one may obtain an expression for $f(x;1,v,w)$, and thus for $f(x;u,v,w)$, via (5). The formula so obtained for $f(x;u,v,w)$ is a somewhat complicated rational function, as are the expressions for the generating functions $f(x;u,1,1)$ and $f(x;1,v,1)$.

The bijection used in the proof of Proposition 4.1 above is seen to preserve both the largest letter and the number of ascents. Thus, in particular, there are also $\binom{n-1+k}{2k}$ members of $\mathcal{S}_n(021,120)$ having k ascents for $0 \le k \le n-1$. We remark that while we were able to find a functional equation comparable to the one in (5) above corresponding to $\mathcal{S}_n(021,120)$, it is possible only to determine the cardinality of $\mathcal{S}_n(021,120)$ from this equation.

Finally, it would be interesting to have direct bijective proofs of Theorems 3.1, 3.2, and 4.5.

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